

GUTs in curved spacetime: running gravitational constants, Newtonian potential and the quantum corrected gravitational equations

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Abstract

The running coupling constants (in particular, the gravitational one) are studied in asymptotically free GUTs and in finite GUTs in curved spacetime, with explicit examples. The running gravitational coupling is used to calculate the leading quantum GUT corrections to the Newtonian potential, which turn out to be of logarithmic form in asymptotically free GUTs. A comparison with the effective theory for the conformal factor —where leading quantum corrections to the Newtonian potential are again logarithmic— is made. A totally asymptotically free $O(N)$ GUT with quantum higher derivative gravity is then constructed, using the technique of introducing renormalization group (RG) potentials in the space of couplings. RG equations for the cosmological and gravitational couplings in this theory are derived, and solved numerically, showing the influence of higher-derivative quantum gravity on the Newtonian potential. The RG-improved effective gravitational Lagrangian for asymptotically free massive GUTs is calculated in the strong (almost constant) curvature regime, and the non-singular De Sitter solution to the quantum corrected gravitational equations is subsequently discussed. Finally, possible extensions of the results here obtained are briefly outlined.

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1 Introduction

The study of the quantum properties of grand unified theories (GUTs) in the presence of a strong curvature is quite an interesting issue, owing to the different applications that it can have in a number of situations. The results of this study are important for a detailed knowledge of the early universe, in particular for an accurate discussion of the known models of inflationary universe (see [1] for a review) and as a guide in the construction of new models of this kind. Moreover, such a study is relevant for a better understanding of quantum effects in the vicinity of a black hole—in particular, for instance, for the calculation of quantum corrections to the black-hole entropy (a recent discussion can be found in [2]). Furthermore, such considerations are of fundamental importance for the estimation of the back reaction effect a quantum field has on the geometry of spacetime (for an earlier discussion, in the free matter case, see [3, 4]).

The renormalization group (RG, see [5] for an introduction) turns out to be very useful in the discussion of GUTs in curved space (see, for example, [6]). The first investigations on these topics, which included the construction of the RG for GUTs in curved spacetime (see [7, 8], and [6] for a complete list of references), have been followed by a lot of activity, where the subject is considered under quite different points of view. Among the different interesting phenomena which are specific of this theory we can count curvature-induced asymptotic freedom [9], asymptotic conformal invariance both in asymptotically-free [7, 6] and in finite GUTs [10], applications of phase transitions of Coleman-Weinberg type in inflationary universes [1], curvature-induced phase transitions [11], and so on.

In the present paper we study the renormalization group properties of GUTs in curved spacetime. We start from asymptotically free GUTs in curved spacetime and write the whole system of RG equations for all the gravitational couplings. Their behaviour is not asymptotically free, of course. Concentrating mainly on the gravitational coupling constant G , we give its general running form in asymptotically free GUTs and provide some explicit examples for the gauge groups $SU(2)$, E_6 (Sect. 2). We also discuss the running couplings for finite GUTs in curved spacetime. The running gravitational coupling is calculated explicitly and it is shown that quantum corrections to G have an exponential form, unlike in asymptotically free GUTs, where they behave power-wise (Sect. 3).

Sect. 4 is devoted to the use of the running gravitational constant for calculations of radiative corrections to the Newton potential. In Sect. 5, in order to study how QG effects may change the qualitative picture obtained in the previous section, we discuss the effective theory of conformal gravity by Antoniadis and Mottola. This theory aims at the description of IR quantum gravity. The running gravitational constant in such a theory looks qualita-

tively similar to that in asymptotically free GUTs, but different from the one in the Einstein theory, where quantum corrections to the Newtonian potential have also been calculated recently [25].

Sect. 6 is devoted to the construction of a totally asymptotically free theory of matter with R^2 -gravity. Considering an $O(N)$ gauge theory with one multiplet of scalars and two multiplets of spinors as matter, and making use of the very interesting technique of introducing potentials in RG-coupling space (similar to a c -function), we explicitly construct the regimes where the total matter-QG theory is asymptotically free (such study is carried out numerically). Then, in Sect. 7 we compare the behaviour of the running gravitational coupling in this matter-QG system with the running of G in the same theory with QG being classical. Sect. 8 is devoted to the study of quantum corrected gravitational equations in the strong curvature regime. The non-singular De Sitter solution of these equations with GUT corrections is discussed, as well as a solution of wormhole type. In the concluding section we summarize our results and outline some possible extension of the same.

2 Asymptotically-free GUTs in curved spacetime and the gravitational coupling constant

Our considerations start from a specific GUT in curved spacetime, given by the following Lagrangian (a multiplicatively renormalizable one, [6])

$$\begin{aligned} L &= L_m + L_{ext}, \\ L_m &= L_{YM} + \frac{1}{2}(\nabla_\mu \varphi)^2 + \frac{1}{2}\xi R\varphi^2 - \frac{1}{4!}f\varphi^4 - \frac{1}{2}m^2\varphi^2 + i\bar{\psi}(\gamma^\mu \nabla_\mu - h\varphi)\psi, \\ L_{ext} &= a_1 R^2 + a_2 C_{\mu\nu\alpha\beta}^2 + a_3 G + a_4 \square R + \Lambda - \frac{1}{16\pi G}R. \end{aligned} \tag{2.1}$$

With an appropriate gauge group, the theory defined by the Lagrangian (2.1) contains gauge fields A_μ , scalars φ and spinors ψ , in some representation of the given gauge group. As usual, the Lagrangian of the external fields must be added to L_m in order to obtain a theory which is multiplicatively renormalizable in curved spacetime [6]. In (2.1) the cosmological constant has been chosen in a specific form which will be convenient in order to facilitate the discussion of the cosmological applications below.

The detailed consideration of the renormalization structure and the RG equations for an asymptotically free GUT of the form (2.1), based on the gauge groups $SU(5)$, $SU(2)$, $O(N)$, E_6 (and some others) can be found in the book [6], where relevant references are listed too. Now, since the theory is multiplicatively renormalizable, the effective Lagrangian that

corresponds to the classical theory (2.1) fulfills the standard RG. We shall assume that the background fields vary slowly with respect to the effective mass of the theory and, therefore, the derivative expansion technique can be used in order to obtain the effective action of the theory. Then, we can restrict our considerations to the terms with only two derivatives with respect to the scalar fields and four derivatives with respect to the purely gravitational terms, and it turns out that the structure of the effective Lagrangian just mimics the structure of the classical Lagrangian (2.1).

The RG equations satisfied by the effective Lagrangian are:

$$\mathcal{D}L_{eff} = \left(\mu \frac{\partial}{\partial \mu} + \beta_i \frac{\partial}{\partial \lambda_i} - \gamma_i \phi_i \frac{\partial}{\partial \phi_i} \right) L_{eff}(\mu, \lambda_i, \phi_i) = 0, \quad (2.2)$$

where μ is the mass parameter, $\lambda_i = (g^2, h^2, f, m^2, \xi, \Lambda, G, a_1, a_2, a_3, a_4)$ is the set of all coupling constants, the β_i are the corresponding beta functions, and $\phi_i = (A_\mu, \phi, \psi)$ are the fields. Note that the dependence on the external gravitational field $g_{\mu\nu}$ is not explicitly shown in L_{eff} .

The solution of Eq. (2.2) by the method of the characteristics gives (for a similar discussion in the case of the RG improved Lagrangian in curved space, see [12]):

$$L_{eff}(\mu, \lambda_i, \phi_i) = L_{eff}(\mu e^t, \lambda_i(t), \phi_i(t)), \quad (2.3)$$

where

$$\begin{aligned} \frac{d\lambda_i(t)}{dt} &= \beta_i(\lambda_i(t)), & \lambda_i(0) &= \lambda_i, \\ \frac{d\phi_i(t)}{dt} &= -\gamma_i(t)\phi_i(t), & \phi_i(0) &= \phi_i. \end{aligned} \quad (2.4)$$

The physical meaning of ((2.3)) and ((2.4)) is that the effective Lagrangian L_{eff} (called sometimes the Wilsonian effective action [5]) is found provided its functional form at some value of t is known (usually the classical Lagrangian serves as boundary condition at $t = 0$). We will come back to the discussion of L_{eff} later on and, for the moment, we shall concentrate on the scaling dependence of the coupling constants, Eq. (2.4). Note that only using the RG-improvement procedure can one also get the non-local effective action, which was discussed in Refs. [35, 36] by direct one-loop calculation.

We consider a typical asymptotically free GUT in curved spacetime. For studying the scaling dependence of the coupling constants, the RG parameter will be chosen to be $t = \ln(\mu/\mu_0)$, as usually, where μ and μ_0 are two different mass scales (see [6] for a rigorous discussion of the RG in curved spacetime). The running coupling constants corresponding

to asymptotically free interaction couplings of the theory have the form

$$\begin{aligned} g^2(t) &= g^2 \left(1 + \frac{B^2 g^2 t}{(4\pi)^2} \right)^{-1}, \quad g^2(0) = g^2, \\ h^2(t) &= \kappa_1 g^2(t), \quad f(t) = \kappa_2 g^2(t), \end{aligned} \quad (2.5)$$

where κ_1 and κ_2 are numerical constants defined by the specific features of the theory under consideration (see [13]-[15] for explicit examples of such GUTs in flat space, and [6] for a review). As one can see, asymptotic freedom ($g^2(t) \rightarrow 0$, $t \rightarrow \infty$) is realized [16], for the gauge coupling and for the running Yukawa and scalar couplings.

The study of this kind of asymptotically free GUT in curved spacetime was been started in Refs. [7, 8] (for a review see [6]), where the formulation in curved space was also developed. Using the results in those works it is easy to show that in such theories (for simplicity, with only one massive scalar multiplet), one obtains

$$\xi(t) = \frac{1}{6} + \left(\xi - \frac{1}{6} \right) \left(1 + \frac{B^2 g^2 t}{(4\pi)^2} \right)^b, \quad m^2(t) = m^2 \left(1 + \frac{B^2 g^2 t}{(4\pi)^2} \right)^b, \quad (2.6)$$

being $\xi(0) = \xi$, $m^2(0) = m^2$, and where for the different models the constant b can be either positive [7, 8, 6], negative [7, 6], or zero [6]. Notice that the constant one-loop running coupling $\xi(t) = \xi$ (i.e. $b = 0$) usually corresponds to supersymmetric asymptotically free GUTs, and also that classical scaling dimensions are not included in the RG equations for couplings with mass dimension, as usually happens in the RG improvement procedure.

Turning now to the gravitational coupling constants [6] (see also [7, 8])

$$\begin{aligned} \frac{da_1(t)}{dt} &= \frac{1}{(4\pi)^2} \left[\xi(t) - \frac{1}{6} \right]^2 \frac{N_s}{2}, \\ \frac{da_2(t)}{dt} &= \frac{1}{120(4\pi)^2} (N_s + 6N_f + 12N_A), \\ \frac{da_3(t)}{dt} &= -\frac{1}{360(4\pi)^2} (N_s + 11N_f + 62N_A), \\ \frac{d\Lambda(t)}{dt} &= \frac{m^4(t)N_s}{2(4\pi)^2}, \\ \frac{d}{dt} \frac{1}{16\pi G(t)} &= -\frac{m^2(t)N_s}{(4\pi)^2} \left[\xi(t) - \frac{1}{6} \right], \end{aligned} \quad (2.7)$$

where N_s , N_f and N_A are, respectively, the number of real scalars, Dirac spinors and vectors and we work in euclidean region. In what follows we shall consider dynamical spacetimes with constant curvature (as the De Sitter space), so that the term in $\square R$ will not appear. As one can see, the behavior of the gravitational couplings $a_2(t)$ and $a_3(t)$ is given by

$a_{2,3}(t) \simeq a_{2,3} + \tilde{a}_{2,3} t$, and it is exactly the same as in the free matter theory with the same field content. It is not influenced by the interaction effects.

The most interesting quantity for us will be the gravitational running coupling constant G . As one can easily find from ((2.7)), it has the form

$$\begin{aligned} G(t) &= G_0 \left\{ 1 - \frac{16\pi N_s G_0 m^2 (\xi - 1/6)}{B^2 g^2 (2b+1)} \left[\left(1 + \frac{B^2 g^2 t}{(4\pi)^2} \right)^{2b+1} - 1 \right] \right\}^{-1} \\ &\simeq G_0 \left\{ 1 + \frac{16\pi N_s G_0 m^2 (\xi - 1/6)}{B^2 g^2 (2b+1)} \left[\left(1 + \frac{B^2 g^2 t}{(4\pi)^2} \right)^{2b+1} - 1 \right] \right\} \\ &\simeq G_0 \left[1 + \frac{N_s}{\pi} G_0 m^2 (\xi - 1/6) t \right], \end{aligned} \quad (2.8)$$

where $t = \ln(\mu/\mu_0)$. As one can see, the matter quantum corrections to the gravitational coupling constant in GUTs are larger than in the free matter theory if $b > 0$.

Let us now give some examples. First of all, we consider the asymptotically free SU(2) model of Ref. [13] with a scalar triplet and two spinor triplets. Then, one can show [7] that $N_s = 3$, $B^2 = 10/3$, $b \simeq 2$. In this model (with the standard choice $g^2 \simeq 0.41$) we get for the first non-trivial correction to the classical G :

$$G(t) \simeq G_0 \left[1 + 0.9549 G_0 m^2 (\xi - 1/6) t \right]. \quad (2.9)$$

As a second interesting example we will consider the asymptotically free E₆ GUT [17] in curved spacetime [18]. This theory contains a 78-plet of real scalars ϕ and two 27-plets of charged scalar, M and N . Assuming that only one mass in the real scalar multiplet is different from zero, and choosing the initial values for the charged scalars to be $\xi_M = \xi_N = 1/6$ we obtain again a $G(t)$ of the same form as (2.8). With the following parameters [18]: $N_s = 78$, $B^2 = 32$, $g^2 = 0.41$, we get

$$\begin{aligned} \xi_\phi(t) &\simeq \frac{1}{6} + 0.97(\xi_\phi - 1/6) \left(1 + \frac{B^2 g^2 t}{(4\pi)^2} \right)^{1/2}, \\ m_\phi^2(t) &\simeq 0.97 m_\phi^2 \left(1 + \frac{B^2 g^2 t}{(4\pi)^2} \right)^{1/2}. \end{aligned} \quad (2.10)$$

Then, we obtain

$$\begin{aligned} G(t) &= G_0 \left\{ 1 + \frac{16\pi N_s G_0 m_\phi^2 0.97^2 (\xi_\phi - 1/6)}{B^2 g^2 (2b+1)} \left[\left(1 + \frac{B^2 g^2 t}{(4\pi)^2} \right)^{2b+1} - 1 \right] \right\} \\ &\simeq G_0 \left[1 + \frac{N_s}{3} 0.97^2 G_0 m_\phi^2 (\xi_\phi - 1/6) t \right] \\ &\simeq G_0 \left[1 + 0.8985 G_0 m_\phi^2 (\xi_\phi - 1/6) t \right]. \end{aligned} \quad (2.11)$$

We thus see that we are able to obtain the general form of the scaling gravitational coupling constant in an asymptotically free GUT in curved spacetime. As we observe from these explicit examples, there exist asymptotically free GUTs in which the matter quantum corrections to the gravitational coupling constants at high energies are bigger than these corrections in the free matter theory. The application of the above results will be discussed later. The gravitational constants for other asymptotically free GUTs can be obtained in a similar way.

3 Finite GUTs in curved spacetime and the gravitational coupling constant

A very interesting class of GUTs in curved spacetime is given by finite GUTs (see, for instance, [19]-[21]). Finite GUTs in curved spacetime were first considered in Ref. [10] (for a review and a complete list of references, see again [6]). Of course, although finite for the interaction couplings in flat space, these theories are not finite in curved spacetime, due to vacuum, mass and $R\varphi^2$ -type divergences. Recently, quite a spectacular development has emerged which concerns $N = 2$ supersymmetric models (in particular, finite) with matter multiplets, through the definition of the exact spectrum [22]. For such theories we have (in the following we consider one-loop finite supersymmetric or non-supersymmetric theories)

$$g^2(t) = g^2, \quad h^2(t) = \kappa_1 g^2, \quad f(t) = \kappa_2 g^2(t), \quad (3.1)$$

and

$$\xi(t) = \frac{1}{6} + \left(\xi - \frac{1}{6}\right) \exp(Cg^2 t), \quad m^2(t) = m^2 \exp(Cg^2 t), \quad (3.2)$$

where the constants κ_1 and κ_2 depend on the specific features of the theory, and where C can be positive, negative or zero [10]. Solving the RG equations for the gravitational coupling constant in such theory, we find

$$\begin{aligned} G(t) &= G_0 \left[1 - \frac{N_s G_0 m^2 (\xi - 1/6)}{2\pi C g^2} (e^{2Cg^2 t} - 1) \right]^{-1} \\ &\simeq G_0 \left[1 + \frac{N_s G_0 m^2 (\xi - 1/6)}{2\pi C g^2} (e^{2Cg^2 t} - 1) \right]. \end{aligned} \quad (3.3)$$

Hence, as one can easily see, GUTs with a positive C are the ones which give the biggest contribution to the gravitational coupling constant, among all finite GUTs.

As an example, one can consider the SU(2) finite model of the first of Refs. [21], with a SU(4) global invariance and the scalar taken in the adjoint representation of SU(2). (Notice

that there is $N = 2$ supersymmetry here, in one of the regimes of finiteness.) In this case, we get that [10] $N_s = 18$ and $C = 24/(4\pi)^2$. As a result,

$$G(t) = G_0 \left[1 + 6\pi \frac{G_0 m^2 (\xi - 1/6)}{g^2} \left(e^{3g^2 t / \pi^2} - 1 \right) \right] \quad (3.4)$$

As one can see, we have power corrections —of the form $\sim \left(\frac{\mu}{\mu_0} \right)^a$ — to the gravitational coupling constant. In a similar way one can also obtain the running gravitational coupling corresponding to other finite GUTs.

4 Quantum GUTs corrections to the Newtonian potential

As an application of the results of the previous discussion, we will consider in this section the quantum corrections to the gravitational potential. From the discussion in Sects. 2 and 3 we know that the typical behavior of the gravitational coupling constant, when taking into account the quantum corrections is

$$G(t) \simeq G_0 \left\{ 1 + \tilde{G}_0 \left[(1 + \tilde{B}t)^{2b+1} - 1 \right] \right\}, \quad (4.1)$$

in asymptotically free GUTs, and

$$G(t) \simeq G_0 \left[1 + \tilde{G}_0 \left(e^{\tilde{C}t} - 1 \right) \right], \quad (4.2)$$

in finite GUTs. Of course, such corrections are too small to be measured explicitly, although they are significantly larger than in free matter theories (due to running). In the above scaling relations, $t = \ln(\mu/\mu_0)$.

It was suggested in Refs. [23, 24] that in the running gravitational and cosmological constants μ/μ_0 ought to be replaced with r_0/r , i.e. that one should change the mass scalar ratio by the inverse ratio between distances. The support for this argument comes from quantum electrodynamics, where the well-known electrostatic potential with quantum corrections can be alternatively obtained from the classical potential by interchanging the classical electric charge with the running one, with $t = \ln(r_0/r)$. Similarly, one can estimate now the GUTs quantum corrections to the gravitational potential. Starting from the classical Newtonian potential

$$V(r) = -\frac{Gm_1 m_2}{r}, \quad (4.3)$$

the classical gravitational constant in (4.3) is to be changed with the running gravitational coupling constant (4.1), (4.2).

As a result, we obtain,

$$V(r) \simeq -\frac{G_0 m_1 m_2}{r} \left[1 + c_1 G_0 m^2 \left(\xi - \frac{1}{6} \right) \ln \frac{r_0}{r} \right], \quad (4.4)$$

in asymptotically free GUTs, and

$$V(r) \simeq -\frac{G_0 m_1 m_2}{r} \left[1 + c_2 G_0 m^2 \left(\xi - \frac{1}{6} \right) \left(\frac{r_0}{r} \right)^{C_2} \right], \quad (4.5)$$

in finite GUTs. Here G_0 is the initial value of G (the value at distance r_0), and C_1 and C_2 are some constants.

In Ref. [25] the gravitational potential has been calculated in the frame of quantum Einsteinian gravity considered as an effective theory (this is unavoidable, owing to its well-known non-renormalizability [26]). It was found that in such theory the leading quantum corrections are proportional to G_0/r^2 . As we see, in an asymptotically free GUT the leading correction is of logarithmic form $\sim \ln(r_0/r)$ while in a finite GUT it is power-like $\sim (r_0/r)^{C_2}$. Hence, from the examples of Sects. 2 and 3 one understands that the GUT quantum corrections to the Newtonian potential can be more important than the corresponding corrections in the effective theory of Einstein's quantum gravity. We now turn to the study of the gravitational constant in some model of quantum gravity.

5 The gravitational constant in the effective theory for the conformal factor

In order to see how quantum gravity (QG) effects change the results of the above discussion, we have to consider some multiplicatively renormalizable QG (with matter). For example, in multiplicatively renormalizable pure higher derivative gravity (for a review, see [6]) the running gravitational coupling has been used to construct the QG corrected Newtonian potential in Ref. [23]. It has been suggested there that the QG corrections in the Newtonian potential (which are also of logarithmic form) might help to solve the dark matter problem and some other cosmological problems [27].

In this section we are going to discuss the running gravitational coupling constant in the effective theory for the conformal factor [28], which presumably describes the infrared phase of QG. The construction of the effective theory for the conformal factor proceeds as follows [28]. One starts from the conformally invariant matter theory (the free theory, for simplicity) in curved spacetime. The standard expression for the conformal anomaly is known (see [29], and for reviews [4, 6]), and it can be integrated on a conformally-flat background (in the

conformal parametrization $g_{\mu\nu} = e^{2\sigma}\bar{g}_{\mu\nu}$, in order to construct the anomaly-induced action. Adding to this action the classical Einsteinian action in the conformal parametrization, we get the effective theory for the conformal factor. In the notations of Ref. [28] and for the flat background $\bar{g}_{\mu\nu} = \eta_{\mu\nu}$, the resulting Lagrangian is

$$L = -\frac{Q^2}{(4\pi)^2}(\Box\sigma)^2 - \zeta \left[2\alpha(\partial_\mu\sigma)^2\Box\sigma + \alpha^2(\partial_\mu\sigma)^4 \right] + \tilde{\gamma}e^{2\alpha\sigma}(\partial_\mu\sigma)^2 - \frac{\lambda}{\alpha^2}e^{4\alpha\sigma}, \quad (5.1)$$

where $\tilde{\gamma} = 3/(8\pi G)$, $\lambda = \Lambda/(8\pi G)$, $\zeta = b + 2b' + 3b''$ and $Q^2/(4\pi)^2 = \zeta - 2b'$, being b, b' and b'' the well-known coefficients of the conformal anomaly [29] (for a recent discussion, see [30]):

$$T_\mu{}^\mu = b \left(C_{\mu\nu\alpha\beta}^2 + \frac{2}{3}\Box R \right) + b'G + b''\Box R. \quad (5.2)$$

Q^2 has been interpreted as the 4-dimensional central charge and α^2 is the anomalous scaling dimension for σ .

Near the infrared (IR) stable fixed point $\zeta = 0$, the theory (5.1) has been argued to describe the IR phase of QG [28]. Near the IR fixed point $\zeta = 0$, the inverse of the running gravitational coupling constant in the IR limit ($t \rightarrow -\infty$) has been calculated in Ref. [31]:

$$\tilde{\gamma}(t) \simeq (-t)^{-1/5} \exp \left[t \left(2 - 2\alpha + \frac{2\alpha^2}{Q_0^2} \right) \right], \quad (5.3)$$

where $Q_0^2 = Q^2(\zeta = 0)$. As a result, we obtain the Newtonian potential with account to the quantum corrections in such a model, as [31]

$$V(r) = -\frac{G_0 m_1 m_2}{r} \left(1 - \frac{\alpha^2}{10Q_0^2} \ln \frac{r_0^2}{r^2} \right). \quad (5.4)$$

As we can see here, essential leading-log corrections to the Newtonian potential appear in this model, what is different from what happens in the case of Einstein gravity.

6 Asymptotic freedom in GUTs interacting with higher derivative quantum gravity

After the above discussion on the running gravitational coupling constant, mainly for GUTs in curved spacetime, our purpose will be to look at its quantum gravitational corrections in frameworks of asymptotically free theories. To be specific, we start from a model with the

following Lagrangian (here we use notations slightly different from those in Sect. 2)

$$\begin{aligned}
\mathcal{L} = & \frac{1}{2\lambda} C_{\mu\nu\alpha\beta} C^{\mu\nu\alpha\beta} - \frac{\omega}{3\lambda} R^2 - \frac{1}{4} G_{\mu\nu}^a G^{a\mu\nu} \\
& + \frac{1}{2} g^{\mu\nu} (\mathcal{D}_\mu \varphi)^i (\mathcal{D}_\nu \varphi)^i + \frac{1}{2} \xi R \varphi^i \varphi^i - \frac{1}{4!} f(\varphi^i \varphi^i)^2 \\
& - \frac{1}{2} m^2 \varphi^i \varphi^i - \frac{R}{16\pi G} + \frac{\Lambda}{8\pi G} \\
& + i\bar{\psi}_p [\gamma^\mu \mathcal{D}_\mu^{pq} - h_i^{pq} \varphi^i] \psi_q.
\end{aligned} \tag{6.1}$$

We will not discuss the problem of unitarity in this theory, which is still open [38] and may be solved, perhaps, only non-perturbatively. We rather consider (6.1) as an effective theory for some unknown consistent QG. This model includes in its gravitational sector higher derivative gravity (with somehow different notations for the gravitational couplings) and in the matter sector an $O(N)$ gauge theory with scalars in its fundamental representation plus n_1 spinor multiplets in the adjoint representation and n_2 spinor multiplets in a fundamental representation. It is known [15, 32] that for some values of N, n_1, n_2 such a flat GUT provides asymptotic freedom for all its coupling constants. Moreover, even taking into account quantum gravity, one can show that asymptotic freedom may survive (see [33]) but under tighter restrictions on the contents of the theory. When studying the asymptotically free regime we look at the region $h^2 \ll g^2$ [33] (so that Yukawa coupling corrections may be dropped). One can show that such a description is consistent under radiative corrections. It is interesting to remark that in the low-energy limit the theory (6.1) leads to the standard Einstein action [10] —similarly to what happens in the pregeometry program [46].

The one-loop beta functions for the theory (6.1) have been calculated in [33, 6]. We will be interested in the study of that system of equations in the $t \rightarrow \infty$ (or high-energy) limit, where it is convenient to make the variable changes [33]

$$\begin{aligned}
\bar{f} &= \frac{f}{g^2}, \\
d\tau &= \kappa g^2(t) dt, \quad \kappa \equiv \frac{1}{(4\pi)^2}.
\end{aligned} \tag{6.2}$$

Further, we will take advantage of the property that, in the limit considered, $\frac{\lambda(t)}{g^2(t)} \rightarrow \frac{b^2}{a^2}$. First, we look at the case of massless couplings only. In these conditions, the total system

of RG equations turns into

$$\begin{aligned}
\frac{dg^2}{d\tau} &= \bar{\beta}_{g^2} = -b^2 g^2, \\
\frac{d\omega}{d\tau} &= \bar{\beta}_\omega = -\frac{b^2}{a^2} \lambda \left[\frac{10}{3} \omega^2 + (5 + a^2) \omega + \frac{5}{12} + \frac{3}{2} N \left(\xi - \frac{1}{6} \right)^2 \right], \\
\frac{d\xi}{d\tau} &= \bar{\beta}_\xi = \left(\xi - \frac{1}{6} \right) \left(\frac{N+2}{3} \bar{f} - \frac{3(N-1)}{2} g^2 \right) \\
&\quad + \frac{b^2}{a^2} \xi \left(-\frac{3}{2} \xi^2 + 4\xi + 3 + \frac{10}{3} \omega - \frac{9}{4\omega} \xi^2 + \frac{5}{2\omega} \xi - \frac{1}{3\omega} \right) \\
\frac{d\bar{f}}{d\tau} &= \bar{\beta}_{\bar{f}} = \frac{N+8}{3} \bar{f}^2 + (b^2 - 3(N-1)) \bar{f} + \frac{9}{4} (N-1) \\
&\quad + \left(\frac{b^2}{a^2} \right)^2 \xi^2 \left(15 + \frac{3}{4\omega^2} - \frac{9\xi}{\omega^2} + \frac{27\xi^2}{\omega^2} \right) \\
&\quad - \frac{b^2}{a^2} \bar{f} \left(5 + 3\xi^2 + \frac{33}{2\omega} \xi^2 - \frac{6}{\omega} \xi + \frac{1}{2\omega} \right) \Big],
\end{aligned} \tag{6.3}$$

where

$$\begin{aligned}
\kappa &= \frac{1}{(4\pi)^2}, \\
a^2 &= \frac{1}{60} (798 + 6N^2 - 5N) + \frac{N}{10} \left(n_2 + \frac{1}{2} (N-1) n_1 \right), \\
b^2 &= \frac{1}{6} (22N - 45) - \frac{4}{3} (n_1 (N-2) + n_2).
\end{aligned} \tag{6.4}$$

It is evident that asymptotic freedom (AF) in the original couplings is determined by the existence of stable fixed points for this new system. Note that the RG equations (6.3) are also very useful for explicit discussions of different forms of effective potential in quantum matter- R^2 -gravity theories [11, 39]

6.1 Without gravity

To make our discussion easier, we start by considering the same situation when gravity is switched off. In these conditions,

$$\begin{aligned}
\frac{dg^2}{d\tau} &= -b^2 g^2, \\
\frac{d\bar{f}}{d\tau} &= \frac{N+8}{3} \bar{f}^2 + (b^2 - 3(N-1)) \bar{f} + \frac{9}{4} (N-1).
\end{aligned} \tag{6.5}$$

This system will be examined using the methods developed in [34], where some potentials in RG coupling-space were introduced, so that their stability properties yield the fixed points

of the original system. One introduces a renormalization group potential $u(g^2, \bar{f})$ such that

$$\begin{aligned}\frac{\partial u}{\partial g^2} &= \frac{dg^2}{d\tau}, \\ \frac{\partial u}{\partial \bar{f}} &= \frac{d\bar{f}}{d\tau}.\end{aligned}\tag{6.6}$$

With the sign convention here used, the existence of stable fixed points amounts to the presence of some sort of maximum for u (if the signs in (6.6) were reversed, we should say minimum instead of maximum). Up to an arbitrary constant, this potential reads

$$u(g^2, \bar{f}) = -\frac{b^2}{2}g^4 + \frac{N+8}{9}\bar{f}^3 + \frac{(b^2 - 3(N-1))}{2}\bar{f}^2 + \frac{9}{4}(N-1)\bar{f}.\tag{6.7}$$

First, we obtain the values of the fixed points, which correspond to the critical points of u in (g^2, \bar{f}) -space. Trivially, $g^2 = 0$, but there are real solutions for \bar{f} only when

$$\Delta \equiv (b^2 - 3(N-1))^2 - 3(N+8)(N-1) \geq 0.$$

If $\Delta \geq 0$, the critical values of \bar{f} are

$$\bar{f}_{\pm} = \frac{-(b^2 - 3(N-1)) \pm \sqrt{\Delta}}{2/3(N+8)}.\tag{6.8}$$

Therefore, the Hessian matrix at these points is

$$\left(\begin{array}{cc} \frac{\partial^2 u}{\partial (g^2)^2} & \frac{\partial^2 u}{\partial g^2 \partial \bar{f}} \\ \frac{\partial^2 u}{\partial \bar{f} \partial g^2} & \frac{\partial^2 u}{\partial \bar{f}^2} \end{array} \right)_{g^2=0, \bar{f}=\bar{f}_{\pm}} = \left(\begin{array}{cc} -b^2 & 0 \\ 0 & \pm\sqrt{\Delta} \end{array} \right).\tag{6.9}$$

Obviously, its eigenvalues are b^2 and $\pm\sqrt{\Delta}$. We need $b^2 > 0$ in order to ensure AF in g^2 . Thus, u can have an extreme only when picking the minus sign, which corresponds to $\bar{f} = \bar{f}_2$, and then that extreme is a local maximum. In such a set-up AF for \bar{f} takes place. By the same argument we conclude that $\bar{f} = \bar{f}_1$ corresponds to a saddle point. The hypothetical case $\Delta = 0$ is rather exceptional; there is only one solution for \bar{f} and the outcome is one vanishing eigenvalue (i.e. the Hessian no longer has maximal rank). Actually, while there is a maximum along the g^2 axis, the potential shows just an inflexion point along the \bar{f} direction, typical of the cubic dependence of u in this variable.

When considering only the \bar{f} -dependent part of u —say $u_{\bar{f}}$ —we realize that it has no global maximum or minimum, because cubic curves are unbounded from above or below. However, if $\Delta > 0$, it has a local minimum (at $\bar{f} = \bar{f}_1$) and a local maximum (at $\bar{f} = \bar{f}_2$).

If $\Delta = 0$, it has an inflexion point at the only zero of $\beta_{\bar{f}}$ and no local extremes. On the contrary, when $\Delta < 0$, $u_{\bar{f}}$ is a monotonic curve without extremes or inflexion points. In the end, it is the presence of the local maximum $\bar{f} = \bar{f}_2$ that accounts for AF.

As we have just seen, the AF scenario is determined by $b^2 > 0$ and $\Delta > 0$. Whether these constraints can be met or not depends, of course, on the particular values of N and of b which, in turn, depends on N , n_1 and n_2 . Assuming $N > 2$, and bearing in mind that n_1, n_2 are integers larger than or equal to zero, from (6.4) one readily finds the condition

$$n_1 \leq 2. \quad (6.10)$$

Then, after numerical examination of Δ for $1 \leq N \leq 10$, $0 \leq n_1 \leq 2$ and $0 \leq n_2 \leq 20$, we find the following combinations yielding AF solution:

| N | n_1 | n_2 |
|-----|-------|----------|
| 7 | 0 | 13 |
| | 1 | 8 |
| | 2 | 3 |
| 8 | 0 | 15,16 |
| | 1 | 9,10 |
| | 2 | 3, 4 |
| 9 | 0 | 17,18,19 |
| | 1 | 10,11,12 |
| | 2 | 3, 4, 5 |
| 10 | 0 | 19,20 |
| | 1 | 11,12,13 |
| | 2 | 3, 4, 5 |

This constitutes an example of (discrete) numerical boundary separating the region where the theory shows a given behaviour for $\tau \rightarrow \infty$.

Let's now see, from another viewpoint, the meaning of our constraints $b^2 > 0$ and $\Delta > 0$ when looking at the couplings themselves. In the present case, we can integrate the RG equations (6.5). For g^2 one has

$$g^2(\tau) = g^2(0)e^{-b^2\tau} \quad (6.11)$$

(we are taking initial conditions so that $\tau = 0$ when $t = 0$). It is plain —as we have said— that $b^2 > 0$ leads to AF in g^2 , while $b^2 < 0$ entails boundless increase of g as τ grows, making perturbation theory no longer valid. With regard to \bar{f} , there are three cases depending on Δ :

1. $\Delta < 0$ (no fixed points)

$$\bar{f}(\tau) = \frac{3}{2(N+8)} \left[\sqrt{-\Delta} \tan \left(\frac{\sqrt{-\Delta}\tau}{2} \right) - (b^2 - 3(N-1)) \right]. \quad (6.12)$$

\bar{f} is periodic in τ , having regularly spaced singularities — analogous to Landau poles.

2. $\Delta = 0$ (one double fixed point)

$$\bar{f}(\tau) = -\frac{(b^2 - 3(N-1))}{2/3(N+8)} - \frac{1}{2/3(N+8)\tau}. \quad (6.13)$$

There is stability in the sense that $\bar{f}(\tau) \rightarrow -\frac{(b^2 - 3(N-1))}{2/3(N+8)}$ as $\tau \rightarrow \infty$, but we still have a pole of this coupling at $\tau = 0$.

3. $\Delta > 0$ (two single fixed points)

$$\bar{f}(\tau) = \frac{(b^2 - 3(N-1) - \sqrt{\Delta})e^{-\sqrt{\Delta}\tau} - (b^2 - 3(N-1) + \sqrt{\Delta})}{2/3(N+8)(1 - e^{-\sqrt{\Delta}\tau})} \quad (6.14)$$

It is clear that now a finite limit exists when τ goes to infinity. In fact,

$$\begin{aligned} \bar{f}(\tau) &\rightarrow -\frac{(b^2 - 3(N-1) + \sqrt{\Delta})}{2/3(N+8)} = \bar{f}_2 \\ \tau &\rightarrow \infty, \end{aligned} \quad (6.15)$$

as should be expected, because we already knew that \bar{f}_2 is the stable fixed point. Now, there can be no doubt that the previous criteria are right and this is indeed the AF region.

6.2 With gravity

Next, we will have to deal with Eqs. (6.3), which offer more serious difficulties than the system (6.5). Since not all the crossed derivatives coincide, we cannot just integrate and find a potential which is a function of all the variables. Instead, we may handle individual potentials for every constant, like in [34]. If we call them $u_{g^2}, u_\omega, u_\xi, u_{\bar{f}}$, one has to require

$$\begin{aligned} \frac{\partial u_{g^2}}{\partial g^2} &= \frac{dg^2}{d\tau}, \\ \frac{\partial u_\omega}{\partial \omega} &= \frac{d\omega}{d\tau}, \\ \frac{\partial u_\xi}{\partial \xi} &= \frac{d\xi}{d\tau}, \\ \frac{\partial u_{\bar{f}}}{\partial \bar{f}} &= \frac{d\bar{f}}{d\tau}. \end{aligned} \quad (6.16)$$

Some possible solutions are

$$\begin{aligned}
u_{g^2} &= -\frac{b^2}{2}g^4, \\
u_\omega &= -\frac{b^2}{a^2}\lambda \left[\frac{10}{9}\omega^3 + \frac{5+a^2}{2}\omega^2 + \left(\frac{5}{12} + \frac{3}{2}N \left(\xi - \frac{1}{6} \right)^2 \right) \omega \right], \\
u_\xi &= \left(\frac{\xi^2}{2} - \frac{1}{6}\xi \right) \left(\frac{N+2}{3}\bar{f} - \frac{3(N-1)}{2}g^2 \right) \\
&\quad + \frac{b^2}{a^2} \left(-\frac{3}{8}\xi^4 + \frac{4}{3}\xi^3 + \frac{3}{2}\xi^2 + \frac{10}{3}\omega\xi - \frac{9}{16\omega}\xi^4 + \frac{5}{6\omega}\xi^3 - \frac{1}{6\omega}\xi^2 \right), \\
u_{\bar{f}} &= \frac{N+8}{9}\bar{f}^3 + \left[\frac{b^2-3(N-1)}{2} - \frac{b^2}{2a^2}\bar{f} \left(5 + 3\xi^2 + \frac{33}{2\omega}\xi^2 - \frac{6}{\omega}\xi + \frac{1}{2\omega} \right) \right] \bar{f}^2 \\
&\quad + \left[\frac{9}{4}(N-1) + \left(\frac{b^2}{a^2} \right)^2 \xi^2 \left(15 + \frac{3}{4\omega^2} - \frac{9\xi}{\omega^2} + \frac{27\xi^2}{\omega^2} \right) \right] \bar{f}
\end{aligned} \tag{6.17}$$

Next, we obtain their critical points (which are zeros of the beta functions (6.3)) by numerical methods. Note e.g. the difference between the \bar{f} -coefficients in $u_{\bar{f}}$ without QG and in the above expression. It is not difficult to realize that the QG contributions may get to change the balance which makes Δ negative or positive, bringing about modifications in the limits of the AF region, as we shall see below.

Once we have the numerical values of all these fixed points, we classify them according to the criterion of the eigenvalues of the matrix

$$\begin{pmatrix}
\frac{\partial^2 u_{g^2}}{\partial (g^2)^2} & \frac{\partial^2 u_{g^2}}{\partial g^2 \partial \omega} & \frac{\partial^2 u_{g^2}}{\partial g^2 \partial \xi} & \frac{\partial^2 u_{g^2}}{\partial g^2 \partial \bar{f}} \\
\frac{\partial^2 u_\omega}{\partial \omega^2} & \frac{\partial^2 u_\omega}{\partial \omega \partial \xi} & \frac{\partial^2 u_\omega}{\partial \omega \partial \bar{f}} & \\
\frac{\partial^2 u_\xi}{\partial \xi^2} & \frac{\partial^2 u_\xi}{\partial \xi \partial \bar{f}} & & \\
\frac{\partial^2 u_{\bar{f}}}{\partial \bar{f}^2} & \frac{\partial^2 u_{\bar{f}}}{\partial \bar{f} \partial \omega} & \frac{\partial^2 u_{\bar{f}}}{\partial \bar{f} \partial \xi} & \frac{\partial^2 u_{\bar{f}}}{\partial \bar{f}^2}
\end{pmatrix} \tag{6.18}$$

taken at the critical points in question (notice that this is not a Hessian matrix, but the standard argument for classifying fixed points leads us to handle it as such). After this numerical work we find that in all cases where we had AF without gravity we also have some solution giving AF with gravity. This is easy to understand by examining the values of the fixed points: of all the solutions found for each (N, n_1, n_2) -combination, there is at least one whose value of ξ tends to be around $1/6$ and whose value of \bar{f} is fairly close to \bar{f}_2 —the AF fixed point without gravity—which gives AF in the presence of QG. In addition, we also find cases where QG makes possible the existence of AF solutions which are banned without QG (as was observed in [33]). These new situations correspond to the values of

| N | n_1 | n_2 |
|-----|-------|-------|
| 7 | 0 | 12 |
| | 1 | 7 |
| | 2 | 2 |
| 8 | 0 | 14 |
| | 1 | 8 |
| | 2 | 2 |
| 9 | 0 | 16 |
| | 1 | 9 |
| | 2 | 2 |
| 10 | 0 | 18 |
| | 1 | 10 |
| | 2 | 2 |

Table 1: Values of N , n_1 and n_2 .

N, n_1, n_2 in the following table and depict a shift in the boundary of the (N, n_1, n_2) -region for which AF existed without QG. More precisely, it is a shift by decreasing the allowed value of n_2 in one unit, as already commented in [33]. Thus, we have shown that the $O(N)$ GUT under discussion, interacting with quantum R^2 -gravity, may be considered a completely asymptotically free theory (for some given field contents of this model). This study has been carried out by introducing a potential in the space of RG couplings, similar to a c -function.

7 Running gravitational coupling in asymptotically free $O(N)$ GUT with quantum R^2 gravity

Now, having at hand the asymptotically free $O(N)$ GUT interacting with quantum R^2 gravity (6.1) we may discuss the behaviour of the gravitational coupling constant. Unlike for GUTs on classical gravitational backgrounds, we cannot analytically solve the RG equations for the gravitational coupling constant. Instead, we have a system of RG equations for the massive couplings m^2 , $\gamma \equiv 1/(16\pi G)$ and Λ , which may be analysed only numerically (after the corresponding study has been done for the massless coupling constant). This system may be explicitly written using the calculations of one-loop counterterms for massive scalars interacting with R^2 gravity in Ref. [37] and the calculation of the scalar γ -function in Ref.

[33]. For the Lagrangian (6.1) we obtain

$$\begin{aligned}
\frac{d\Lambda}{dt} &= \beta_\Lambda = \kappa \left[\frac{N}{4}\gamma m^4 + \Lambda N m^2 \left(\xi - \frac{1}{6} \right) + \frac{\lambda\Lambda}{2} \left(\frac{43}{3} + \frac{20}{3}\omega + \frac{1}{6\omega} \right) \right. \\
&\quad \left. + \lambda^2 \gamma \left(\frac{5}{4} + \frac{1}{16\omega^2} \right) \right], \\
\frac{d\gamma}{dt} &= \beta_\gamma = -\kappa \left[\lambda \gamma \left(\frac{10}{3}\omega - \frac{1}{4\omega} - \frac{13}{6} \right) + N m^2 \left(\xi - \frac{1}{6} \right) \right], \\
\frac{dm^2}{dt} &= \beta_{m^2} = \kappa \left[m^2 \left(\frac{N+2}{3}f - \frac{3}{2}(N-1)g^2 \right) \right. \\
&\quad \left. + \lambda m^2 \left(-\frac{43}{6} - \frac{5}{12\omega} + 3\frac{\xi}{\omega} - \frac{9}{4\omega}\xi^2 - \frac{3}{2}\xi^2 \right) \right. \\
&\quad \left. + \lambda^2 \xi \gamma \left(5 + \frac{1}{4\omega^2}\xi^2 + \frac{33}{2\omega^2} \right) \right].
\end{aligned} \tag{7.1}$$

7.1 Without gravity

Ignoring all the QG pieces, and performing the changes (6.2), we are posed with a system of differential equations consisting of (6.5) plus

$$\begin{aligned}
\frac{d\Lambda}{d\tau} &= \frac{N}{4} \frac{m^4}{g^2} \gamma + \Lambda N \frac{m^2}{g^2} \left(\xi - \frac{1}{6} \right), \\
\frac{d\gamma}{d\tau} &= -N \frac{m^2}{g^2} \left(\xi - \frac{1}{6} \right), \\
\frac{dm^2}{d\tau} &= m^2 \left[\frac{N+2}{3} \bar{f} - \frac{3}{2}(N-1) \right].
\end{aligned} \tag{7.2}$$

It is possible to estimate the type of γ solution to this system in the large- τ regime. Considering the most interesting case i.e. $\Delta > 0$, —and therefore $\bar{f}(\tau)$ given by (6.14)— we take approximations of the type $e^{\sqrt{\Delta}\tau} - 1 \sim e^{\sqrt{\Delta}\tau}$ and, using (6.11), arrive at

$$\begin{aligned}
m^2(\tau) &\sim m^2(0) e^{-\mathcal{B}\tau/2}, & \mathcal{B} &= \frac{N+2}{N+8} [b^2 - 3(N-1)] + 3(N-1), \\
\gamma(\tau) &\sim \gamma(0) + \gamma_1 \left(1 - e^{(b^2 - \mathcal{B}/2)\tau} \right), & \gamma_1 &= \frac{N \left(\xi - \frac{1}{6} \right) m^2(0)}{b^2 - \mathcal{B}/2} \frac{1}{g^2(0)}.
\end{aligned} \tag{7.3}$$

Actually, \mathcal{B} is positive whenever we are in the settings of AF for both g^2 and \bar{f} —i.e. (N, n_1, n_2) combinations in the first table— while $b^2 - \mathcal{B}/2$ is negative in these same cases. As we see, under the present assumptions $\gamma(\tau)$ would tend asymptotically to a constant value of $\gamma(0) + \gamma_1$ as $\tau \rightarrow \infty$. Undoing now the variable change (6.2), we may put $\tau = \frac{1}{b^2} \log(1 + \kappa b^2 g^2(0)t)$, and writing $G(t)$ in terms of $\gamma(t)$ we are left with

$$G(t) = G(0) \left\{ 1 - 16\pi G(0) \gamma_1 \left[(1 + \kappa b^2 g^2(0)t)^{1 - \frac{\mathcal{B}}{2b^2}} - 1 \right] \right\}^{-1}, \tag{7.4}$$

where, obviously, $G(0) = 1/(16\pi\gamma(0))$. This expression coincides with (2.8) after making the notational replacements $N_s \rightarrow N$, $B^2 \rightarrow b^2$, $2b + 1 \rightarrow 1 - \frac{B}{2b^2}$. The above remarked asymptotic behaviour is clearly appreciated in Fig. 1.

7.2 With Gravity

We can solve numerically the whole system of differential equations and examine the asymptotically free regime of its solutions. By plotting the gravitational running coupling, we obtain the curves a and b in Fig. 2.

8 Gravitational field equations with GUT quantum corrections in curved spacetime

Let's now turn to some other application of the running coupling constants in curved space-time, namely to the RG-improved effective Lagrangian (2.3). We will work for simplicity on the purely gravitational (almost constant) background, supposing that for all quantum fields we have zero background. In actual one-loop calculations, it turns out that, working in configuration space, the RG parameter t is typically of the form

$$t \simeq \frac{1}{2} \log \frac{c_1 R + c_2 m^2}{\mu^2}, \quad (8.1)$$

where c_1 , c_2 are some numerical constants, and m^2 is the effective mass of the theory. In the model under discussion we have a few different masses, so there is no unique way of choosing only one functional form for t [12, 44]. Hence, we will consider the regime of strong curvature when curvature is dominant in (8.1). Then, the natural choice of t in the RG-improved Lagrangian is (see also [12, 9])

$$t = \frac{1}{2} \log \frac{R}{\mu^2}. \quad (8.2)$$

Note that such a regime may lead to curvature-induced asymptotic freedom [9]. In this regime we obtain that expression (2.3) gives the leading-log approach to the whole perturbation series.

$$S_{\text{RG}} = \int d^4x \sqrt{-g} \left\{ \Lambda(t) - \frac{1}{16\pi G(t)} R + a_1(t) R^2 + a_2(t) C_{\mu\nu\alpha\beta} C^{\mu\nu\alpha\beta} + a_3(t) G \right\}. \quad (8.3)$$

We will consider only asymptotically free GUTs, where $\Lambda(t)$, $G(t)$ are given in Sect. 2, $t = \frac{1}{2} \log \frac{R}{\mu^2}$, $a_{2,3}(t) = a_{2,3} + \tilde{a}_{2,3}t$ (see (2.7)). Note, however, that equation (8.3) is very

general. In particular, it has a similar form in the asymptotically free GUTs with R^2 -gravity of Sects. 6,7 in strong curvature regime (however, the t -dependence of the running couplings $\Lambda(t), \dots, a_3(t)$ is of course different). Notice that the technique developed in Ref. [41] may be very useful for the explicit solution of quantum corrected gravitational field equations for non-constant curvature.

Working in constant curvature space

$$R_{\mu\nu} = \frac{g_{\mu\nu}}{\beta^2}, \quad R = \frac{4}{\beta^2} \quad (8.4)$$

we may rewrite (8.3) expanding the coupling constants up to linear t terms as follows

$$\begin{aligned} S_{\text{RG}} &= \text{const} \times \beta^4 \left\{ \Lambda + \tilde{\Lambda}t - \frac{1}{16\pi} \left(\frac{1}{G} + \tilde{G}t \right) \frac{4}{\beta^2} \right. \\ &\quad \left. + (a_1 + \tilde{a}_1 t) \frac{16}{\beta^4} + (a_3 + \tilde{a}_3 t) \frac{8}{3\beta^4} \right\}, \\ t &= \frac{1}{2} \log \frac{4}{\beta^2 \mu^2}, \end{aligned} \quad (8.5)$$

where the explicit form of the constants $\tilde{\Lambda}$, \tilde{G} , \tilde{a}_1 , \tilde{a}_3 is evident from (2.7).

Now one can write the field equation which fixes β^2 in terms of the theory parameters:

$$\begin{aligned} \frac{\partial S_{\text{RG}}}{\partial \beta^2} &= \text{const} \times \left\{ \left[-\frac{\tilde{\Lambda}}{2} + 2(\Lambda + \tilde{\Lambda}t) \right] \beta^2 - \frac{1}{4\pi} \left[-\frac{\tilde{G}}{2} + \left(\frac{1}{G} + \tilde{G}t \right) \right] \right. \\ &\quad \left. - \left(16\tilde{a}_1 + \frac{8}{3}\tilde{a}_3 \right) \frac{1}{2\beta^2} \right\} = 0. \end{aligned} \quad (8.6)$$

One can see that there exists a real root of (8.6) for most asymptotically free GUTs (for a massless theory, see also discussion in Ref. [9]). Hence, we have got a non-singular universe with a metric of the form

$$ds^2 = a^2(\eta) \left(d\eta^2 - \frac{dx^2 + dy^2 + dz^2}{\left(1 + \frac{kx^2}{12}\beta^2 \right)^2} \right) \quad (8.7)$$

where $k = 1, 0, -1$ for a closed, asymptotically open and open universe, respectively, and

$$a(\eta) = \frac{\sqrt{k}}{2} \left\{ \text{tg} \left(-\frac{\sqrt{k}\eta}{2\sqrt{3}\beta} \right) + \text{tg}^{-1} \left(-\frac{\sqrt{k}\eta}{2\sqrt{3}\beta} \right) \right\} \quad (8.8)$$

as the solution of the gravitational field equations with quantum GUT corrections. The GUTs quantum corrections are here present in the form of β , which is defined by GUT parameters from (8.6). In terms of physical time T , the scale factor may be put as

$$a(T) = \frac{\sqrt{3}\beta}{2} \left[e^{T/(\sqrt{3}\beta)} + k e^{-T/(\sqrt{3}\beta)} \right]. \quad (8.9)$$

Note that such a non-singular inflationary type (De Sitter) solution in gravity theories with higher derivative terms (induced by quantum matter) has been discussed in Refs. [4, 9, 40, 41] in different contexts (for a very recent discussion on non-singular cosmologies in higher-derivative theories, see [42]).

Let us consider the closed universe ($k = 1$) in (8.7). In this case, for some choices of the theory parameters one can find an imaginary β as the solution of Eq. (8.6). Similarly to what happens in papers [43], such a solution may be interpreted as a Lorentzian wormhole which connects two De Sitter universes. However, this wormhole, that for $|\beta^2| \sim L_{Pl}^2$ has a ‘mouth’ of the Planck size, results from quantum GUT corrections. From another viewpoint, working with (8.9) at $k = 1$ and imposing the initial conditions $a(0) = R_0$, $\dot{a}(0) = 0$ we get

$$a(T) = \sqrt{3}\beta \cosh\left(\frac{T}{\sqrt{3}\beta}\right). \quad (8.10)$$

Some analysis of this solution is given in [45]. For example, when the effective cosmological constant is zero, the solution (8.10) corresponds to a closed universe connected through a wormhole to flat space. In a similar fashion one can construct the quantum corrected gravitational equations in other regimes and study their solutions.

9 Discussion

In the present paper we have discussed the running of the gravitational coupling constant in asymptotically free GUTs in curved spacetime, in the effective theory for the conformal factor and in asymptotically free R^2 -gravity interacting with an $O(N)$ GUT. The running gravitational coupling constant has been used to calculate the leading quantum corrections to the Newtonian potential. These corrections have logarithmic form in asymptotically free GUTs and in the effective theory for the conformal factor, and power-like form (but of different nature from that in Einsteinian gravity) in finite GUTs. In R^2 -gravity with $O(N)$ GUT, the behaviour of the gravitational coupling constant is numerically analysed. Its decay rate gets higher as the value of $m^2(0)$ is raised.

The running coupling constants are also necessary in other respects, particularly in the RG-improvement procedure. We have found the RG-improved effective gravitational Lagrangian in the regime of strong constant curvature, and have discussed the non-singular De Sitter solution of the corresponding quantum corrected gravitational field equations. The present technique is quite general and can be applied in various situations, in particular for the construction of RG-improved non-local gravitational Lagrangian, what we plan to discuss elsewhere.

The other interesting field where the running gravitational constants calculated in this paper play an important role is in the quantum corrections to the Hawking-Bekenstein black hole entropy. The black hole entropy (Bekenstein-Hawking formula) has the following form

$$S = \frac{A}{4G},$$

where A is the surface area of the event horizon and G the

gravitational constant. It has been suggested by Susskind et al. [2] that in the brick wall approach (e.g. the 't Hooft cut-off regularization [47]) the quantum corrections to the black-hole entropy can be absorbed in the above formula as a simple renormalization of the gravitational constant G . In this respect, taking into account the results of our study we are led to conjecture that in the case of dimensional regularization (where only logarithmic divergences are important) the quantum corrections to the Bekenstein-Hawking formula are precisely given by the standard renormalization of G , as discussed above in the present paper —plus some less essential contributions from the higher-derivative terms in (2.1). This gives further relevance to our calculations in the paper, since then, in order to take into account GUT contributions to the black hole entropy one just has to use $G(t)$ instead of G in the formula for the entropy.

Note that the quadratic terms in the lagrangian contribute the black hole entropy already at tree level as shown in Refs [48, 41, 49]. For the charged black hole they give corrections like

$$S = \frac{A_H}{4G} + \frac{16\pi^2(3a_2 + 2a_3)Q^2}{A_H} + \dots$$

From the results of our paper we can further infer that the one-loop corrections will have the form of this equation but with all the coupling constants being now a function of t . We hope to return to this question in near future.

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Figure captions

Fig. 1. Running gravitational coupling $G(t) = 1/(16\pi\gamma(t))$ obtained by numerical integration of the full system made of Eqs. (6.3) (written in terms of the original RG parameter t of Ref. [33]) and Eqs. (7.1), for $N = 7$, $n_1 = 0$, $n_2 = 13$. The initial values are $g^2(0) = 0.1$, $\omega(0) = 0.1$, $\xi(0) = 0$, $f(0) = 0.5$, $\Lambda(0) = 0.1$, $\gamma(0) = 0.1$, $m^2(0) = 0.1$ and $\lambda(0) = 0$, i.e. initially there is no quantum gravity. The asymptotic tendency of $G(t)$ towards a constant value is clear.

Fig. 2. $G(t)$ in presence of QG with the same initial conditions as in Fig. 1, except for: a) $\lambda(0) = 0.5$ and $m^2(0) = 0$, b) $\lambda(0) = 0.5$ and $m^2(0) = 0.5$. The running gravitational constant is quickly decreasing.

Fig. 1

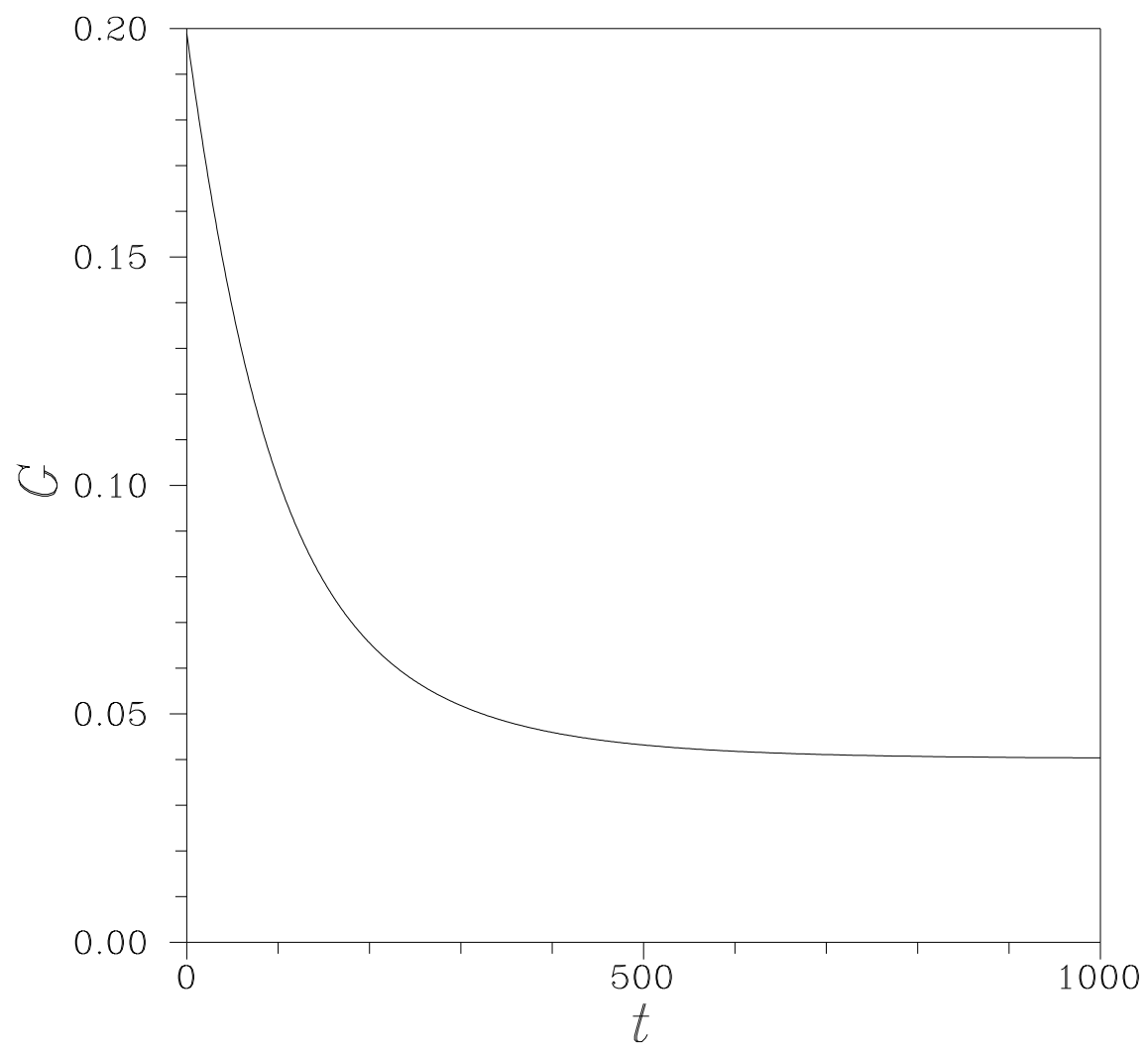


Fig. 2

